Additional Vector Analysis

Airocéan*

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Proof 1 of dyatic vector.

$$\begin{aligned}\vec{a}\times(\vec{b}\cdot\vec{c})\vec{d} =& \vec{a}\times\vec{d}(\vec{b}\cdot\vec{c})\\ =& \vec{e}(\vec{b}\cdot\vec{c})\\ =& (\vec{e}_2-\vec{e}_1)[(\vec{b}_2-\vec{b}_1)\cdot\vec{c}]\\ =& [\vec{e}_2\vec{b}_2-\vec{e}_2\vec{b}_1-\vec{e}_1\vec{b}_2+\vec{e}_1\vec{b}_1]\cdot\vec{c}.\end{aligned} \tag{1}$$

Proof of $\vec{k}=-(\vec{b}\cdot\nabla)\vec{b}$. The basic:

$$\begin{aligned}\vec{b} &= \vec{\imath}b_x+\vec{\jmath}b_y+\vec{k}b_z,\\ \nabla &= \vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}.\end{aligned} \tag{2}$$

For $\vec{b}\cdot\nabla$ as a united operator:

$$\vec{b}\cdot\nabla = b_x\frac{\partial}{\partial x}+b_y\frac{\partial}{\partial y}+b_z\frac{\partial}{\partial z}. \tag{3}$$

The curvature $\vec{k}$ (not the same $\vec{k}$ as the unit vector) can be defined as $\mathrm{d}\vec{b}/\mathrm{d}l$ (which is very straightforward to understand):

$$\frac{\mathrm{d}\vec{b}}{\mathrm{d}l} = \frac{\vec{\imath}\mathrm{d}b_x+\vec{\jmath}\mathrm{d}b_y+\vec{k}\mathrm{d}b_z}{\mathrm{d}l}. \tag{4}$$

Multiple the operator $\vec{b}\nabla$ and $\vec{b}$, focus on the first term (since the second and the third term use the same approach):

$$\vec{\imath}\left(b_x\frac{\partial b_x}{\partial x}+b_y\frac{\partial b_x}{\partial y}+b_z\frac{\partial b_x}{\partial z}\right). \tag{5}$$

We have:

$$\begin{aligned}\mathrm{d}b_x &= \frac{\partial b_x}{\partial x}\mathrm{d}x+\frac{\partial b_x}{\partial y}\mathrm{d}y+\frac{\partial b_x}{\partial z}\mathrm{d}z,\\ \frac{\mathrm{d}b_x}{\mathrm{d}l} &= \frac{\partial b_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}l}+\frac{\partial b_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}l}+\frac{\partial b_x}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}l}.\end{aligned} \tag{6}$$

Using the definition of the unit vector $\vec{b}$ and a very easy triangle relation, we have:

$$\frac{\mathrm{d}x}{\mathrm{d}l} = b_x = \frac{b_x}{|b|}, \tag{7}$$

combine equation **??** and equation **??**, it is very easy to get:

$$b_x\frac{\partial b_x}{\partial x}+b_y\frac{\partial b_x}{\partial y}+b_z\frac{\partial b_x}{\partial z}, \tag{8}$$

which happens to be the norm of the curvature vector $\vec{k}$ in the $\vec{\imath}$ direction, which is shown in equation **??**. This is also why the equation should be written at the form of $\vec{k}=-(\vec{b}\cdot\nabla)\vec{b}$, instead of ~~$\vec{k}=-(\vec{b}\nabla)\cdot\vec{b}$~~ or something else. This will soon be proven next.

Proof of why the Taylor Formula of a vector should be written at the form of $(\vec{r}\cdot\nabla)\vec{B}_0$ instead of $\vec{r}(\nabla\cdot\vec{B}_0)$.

The ordinary Taylor Formula is written as:

$$f(x)=\frac{f\left(x_0\right)}{0!}+\frac{f'\left(x_0\right)}{1!}\left(x-x_0\right)+\ldots+\frac{f^{(n)}\left(x_0\right)}{n!}\left(x-x_0\right)^n+\ldots \tag{9}$$

For a multi-variables function:

$$\begin{aligned}f(x,y)=f(x_0,y_0)&+\frac{(\partial f/\partial x)^{(1)}}{1!}(x-x_0)+\ldots\\ &+\frac{(\partial f/\partial y)^{(1)}}{1!}(y-y_0)+\ldots\end{aligned} \tag{10}$$

And,

$$\vec{B}[B_x(x,y,z),B_y(x,y,z),B_z(x,y,z)] = \vec{\imath}B_x+\vec{\jmath}B_y+\vec{k}B_z \tag{11}$$

Using Taylor Formula,

$$\begin{aligned}B_x(x,y,z)=B_x(x_0,y_0,z_0)&+\frac{\partial B_x}{\partial x}(x-x_0)+\ldots\\ &+\frac{\partial B_x}{\partial y}(y-y_0)+\ldots\\ &+\frac{\partial B_x}{\partial z}(z-z_0)+\ldots\end{aligned} \tag{12}$$

And for $\vec{r}(\nabla\cdot\vec{B})$,

$$\vec{r}(\nabla\cdot\vec{B}) = (\vec{\imath}x+\vec{\jmath}y+\vec{k}z)\cdot\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right), \tag{13}$$

and,

$$(\vec{r}\cdot\nabla)\vec{B} = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\cdot(\vec{\imath}B_x+\vec{\jmath}B_y+\vec{k}B_z). \tag{14}$$

Which one meets the demand of equation **??** is extremely obvious. The proof can always be used in the first order approximation. Like when we try to calculate the magnetic drift in plasma physics.

Proof of $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-(\nabla\cdot\nabla)\vec{A}$. First,

$$\begin{aligned}\nabla\times\vec{B} &= \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ Bx & By & Bz\end{vmatrix}\\ &= \left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\vec{\imath}+\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\vec{\jmath}+\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\vec{k}\end{aligned} \tag{15}$$

$$\nabla\times(\nabla\times\vec{A}) = \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial A_z}{\partial y}-\dfrac{\partial A_y}{\partial z} & \dfrac{\partial A_x}{\partial z}-\dfrac{\partial A_z}{\partial x} & \dfrac{\partial A_y}{\partial x}-\dfrac{\partial A_x}{\partial y}\end{vmatrix}. \tag{16}$$

$$\nabla(\nabla\cdot\vec{A}) = \left(\vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right), \tag{17}$$

$$(\nabla\cdot\nabla)\vec{A} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\vec{\imath}A_x+\vec{\jmath}A_y+\vec{k}A_z). \tag{18}$$

Then, it is quite easy to use preliminary school calculating method to see their mathematical relation. **All proofs above can be better written in a tensor form.**

$$\nabla\cdot\boldsymbol{A} = \sum_i\frac{\partial A_i}{\partial x_i} = \partial_i A^i. \tag{19}$$

$$\begin{aligned}\nabla\times\boldsymbol{A} &= \sum_i\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right)\hat{e}_i\\ &= \sum_{ijk}\varepsilon_{ijk}\frac{\partial A_j}{\partial x_i}\hat{e}_k\\ &= \varepsilon^{ijk}\partial_i A_j = \varepsilon^{kij}\partial_i A_j.\end{aligned} \tag{20}$$

*airocean@mail.ustc.edu.cn, a@airocean.cn, airocean@foxmail.com, http://airocean.cn